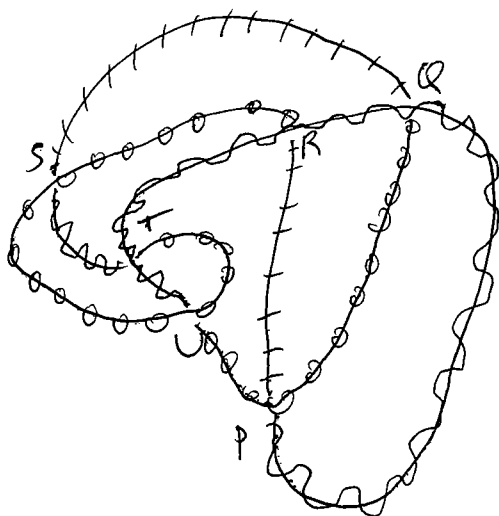


1. a)



Similar
to
Coursework

10

all one colour at P $\Rightarrow$ all one colour at Q $\Rightarrow$ all one colour at R $\Rightarrow$ all one colour at S $\Rightarrow$ all one colour at T $\Rightarrow$ all one colour at U, but at least 3 colours must be used. #

Suppose 3 different colours are at P as in diagram. Then ~~the knot is not tricolourable~~ R is coloured as in diagram. ~~the knot is not tricolourable~~ #

Therefore the knot has 2 colours. #

Therefore the knot is not tricolourable.

b) Suppose both strands have same colour. Then

insures knot is still tricoloured after the move. #



Suppose the strands have 2 different colours. Then



insures the knot is still tricoloured after the move.

Done
in class

(2) similar to theorem proved in class

9

$$7 \quad W \left(\text{diagram 1} \right) = W \left(\text{diagram 2} \right) + | - | - | - | \quad 2 \text{ marks}$$

$$= W \left(\text{diagram 3} \right) - 2 \quad 2 \text{ marks}$$

$$\neq W \left(\text{diagram 4} \right) \quad 1 \text{ mark}$$

so the Queen Mary move is not invariant for W with the. 2 marks

2. a) Reidemeister Moves II and III are
3 invariant for Kauffman Bracket.

done in class

3 marks

10 b) $B \left(\text{diagram} \right) = B \left(\text{diagram} \right)$
 $= B \left(\text{diagram} \right) x^{-1} + B \left(\text{diagram} \right) x$

because
Reid move II
was
performed

4 marks

$= B \left(\text{diagram} \right) \left[x^{-1} + x \left(-x^2 - x^{-2} \right) \right]$

$= B \left(\text{diagram} \right) (-x^3)$

or they can use
fact its Reid move I

4 marks

$= (x^{-7} - x^{-3} - x^5) (-x^3)$

$= -x^{-4} + 1 + x^8$

similar to calculations
done in exercises

2 marks

9 c) $W \left(\text{diagram} \right) = -1 -1 -1 -1 +1 = -4$

4 marks

$J(x) = (-x)^{34} B = (-x)^{-12} (x^{-4} + 1 + x^8)$
 $= -x^{-16} + x^{-12} + x^{-4}$

2 marks

similar to
exercises

2 marks

letting $t = x^4$, $J(t) = -t^{-4} + t^{-3} + t^{-1}$

4 d) $J_R(t) = J(t^{-1})$. for the knot in b) ~~$J(t)$~~

2 marks

in b) $J(t^{-1}) = -t^4 + t^3 + t \neq J(t)$. Therefore

$J_R(t) \neq J(t)$ and the knot is chiral.

2 marks

done in class

$$3. a) \tau = \frac{\dot{\gamma}}{\|\dot{\gamma}\|}, \quad \beta = \tau \times \dot{\gamma}, \quad \beta = \frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|}$$

5

If γ is unit speed $v = \frac{\ddot{\gamma}}{\|\ddot{\gamma}\|}$

3 marks

done in class

(4)

2 marks

$$4) \dot{\gamma} = \left(-\frac{4}{5} \sin t, -\cos t, \frac{3}{5} \sin t \right)$$

similar to exercises

$$14 \quad \|\dot{\gamma}\|^2 = \frac{16}{25} \sin^2 t + \cos^2 t + \frac{9}{25} \sin^2 t$$

$$\Rightarrow \sin^2 t + \cos^2 t = 1 \Rightarrow \gamma \text{ unit speed}$$

4 marks

$$\Rightarrow \tau = \dot{\gamma} = \left(-\frac{4}{5} \sin t, -\cos t, \frac{3}{5} \sin t \right)$$

2 marks

$$\ddot{\gamma} = \left(-\frac{4}{5} \cos t, \sin t, \frac{3}{5} \cos t \right)$$

$$\|\ddot{\gamma}\|^2 = \frac{16}{25} \cos^2 t + \sin^2 t + \frac{9}{25} \cos^2 t$$

$$= \cos^2 t + \sin^2 t = 1$$

$$\Rightarrow v = \ddot{\gamma} = \left(-\frac{4}{5} \cos t, \sin t, \frac{3}{5} \cos t \right)$$

2 marks

$$\beta = \tau \times v = \begin{pmatrix} -\frac{3}{5} \sin t - \frac{3}{5} \cos^2 t, \frac{12}{25} \sin t \cos t - \frac{12}{25} \sin t \cos t, \\ -\frac{6}{5} \sin^2 t - \frac{4}{5} \cos^2 t \end{pmatrix}$$

$$= \left(\frac{3}{5}, 0, -\frac{4}{5} \right) \left(-\frac{1}{5}, 0, -\frac{4}{5} \right)$$

2 marks

$$K = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3} = \frac{\|\tau \times v\|}{1} = \|\beta\| = \frac{\sqrt{\frac{9}{25} + 0 + \frac{16}{25}}}{1} = \frac{\sqrt{5}}{5}$$

2 marks

$$T = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} = \beta \cdot \left(\frac{4}{5} \sin t, \cos t, -\frac{3}{5} \sin t \right) = \left(-\frac{12}{25} + \frac{12}{25} \right) \sin t = 0$$

2 marks

c) we know

$$\dot{\beta} = -\tau \gamma = 0$$

2 marks

Thus, $\frac{d}{dt}(\gamma(H) \cdot \beta)$

$$= \dot{\gamma} \cdot \beta + \gamma \cdot \dot{\beta}$$

2 marks

$$= \dot{\gamma} \cdot \beta$$

$$= \dot{\gamma} \cdot \left(\frac{\dot{\gamma} \wedge \ddot{\gamma}}{\|\dot{\gamma} \wedge \ddot{\gamma}\|} \right) \text{ by definition of } \beta$$

$$= 0 \quad \text{because } \dot{\gamma} \wedge \ddot{\gamma} \text{ is } \perp \text{ to } \dot{\gamma}$$

2 marks

(6)

4 a) A curve $\gamma: (a,b) \rightarrow \mathbb{R}^3$ is regular
 if $\dot{\gamma}(t) \neq 0 \quad \forall t \in (a,b)$

from dens

3 marks

10 b) $\dot{\gamma} = (-a \sin t, b \cos t, 0)$
 $\ddot{\gamma} = (-a \cos t, -b \sin t, 0)$

$$\|\dot{\gamma}\|^2 = a^2 \sin^2 t + b^2 \cos^2 t$$

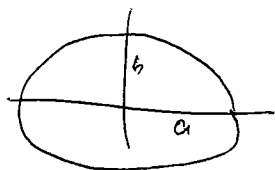
2 marks

$$K_S \hat{k} = \frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma}\|^3}$$

$$\dot{\gamma} \times \ddot{\gamma} = a b \sin^2 t - (-a b \cos^2 t) = a b$$

$$\Rightarrow K_S = \frac{a b}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}$$

3 marks



curve is ellipse which "turns" once

Therefore $2\pi(1) = \int_0^{2\pi} K_S \|\dot{\gamma}\| dt$

$$2\pi = \int_0^{2\pi} \frac{a b}{a^2 \sin^2 t + b^2 \cos^2 t} dt$$

similar
to exercises

5 marks

12 c) $T = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot (C\dot{\gamma})}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} = 0$

because
 $\dot{\gamma} \times \ddot{\gamma} \perp \dot{\gamma}$

5 marks

$$K_S \hat{k} = \dot{\gamma} \times \ddot{\gamma} \Rightarrow \dot{K}_S \hat{k} = \dot{\gamma} \times \ddot{\gamma} + \dot{\gamma} \times \ddot{\gamma}$$

$$= 0 + \dot{\gamma} \times (C\dot{\gamma})$$

$$= 0$$

5 marks

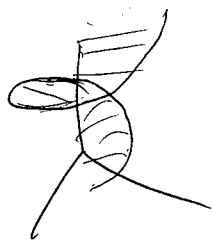
$\Rightarrow K_S$ constant $\Rightarrow$ part of circle.

unseen

2 marks

(or differentiate $K^2 = \|\dot{\gamma} \times \ddot{\gamma}\|^2$)

S. 9)



like
exercises
& examples
in class

(7)

$$\sigma_u = (-v \sin u, v \cos u, \lambda)$$

$$\sigma_v = (\cos u, \sin u, 0)$$

$$\begin{aligned} \sigma_u \times \sigma_v &= \left(\cancel{-v \sin^2 u}, \cancel{v \cos^2 u}, \lambda \cos u \right. \\ &\quad \left. - \lambda \sin u, \lambda \cos u, -v \sin^2 u - v \cos^2 u \right) \\ &= (-\lambda \sin u, \lambda \cos u, -v) \end{aligned}$$

$$\|\sigma_u \times \sigma_v\| = \sqrt{\lambda^2 (\sin^2 u + \cos^2 u) v^2} = \sqrt{\lambda^2 + v^2} > 0 \quad (\text{because } \lambda > 0)$$

$\Rightarrow N_\sigma$ is regular

5 marks

$$b) \quad E = \sigma_u \cdot \sigma_u = v^2 (\sin^2 u + \cos^2 u) + \lambda^2 = v^2 + \lambda^2$$

$$F = \sigma_u \cdot \sigma_v = (-v \sin u \cos u + v \cos u \sin u) = 0$$

$$G = \sigma_v \cdot \sigma_v = \cos^2 u + \sin^2 u = 1$$

$$F_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} v^2 + \lambda^2 & 0 \\ 0 & 1 \end{pmatrix}$$

5 marks

$$N_\sigma = \left(-\frac{\lambda}{\sqrt{\lambda^2 + v^2}} \sin u, \frac{\lambda}{\sqrt{\lambda^2 + v^2}} \cos u, -\frac{v}{\sqrt{\lambda^2 + v^2}} \right)$$

$$\sigma_{uu} = (-v \cos u, -v \sin u, 0) \quad \sigma_{uv} = (-\sin u, \cos u, 0)$$

$$\sigma_{vv} = (0, 0, 0)$$

$$L = \sigma_{uu} \cdot N_\sigma = 0 \quad M = \sigma_{uv} \cdot N_\sigma = \frac{\lambda}{\sqrt{\lambda^2 + v^2}}, \quad N = \sigma_{vv} \cdot N_\sigma = 0$$

$$F_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} 0 & \frac{\lambda}{\sqrt{\lambda^2 + v^2}} \\ \frac{\lambda}{\sqrt{\lambda^2 + v^2}} & 0 \end{pmatrix}$$

5 marks

c) The principle curvatures solve

$$\det(F_{II} - \lambda F_I) = 0$$

$$= \det \begin{pmatrix} -(v^2 + \lambda^2)x & -\frac{\lambda}{\lambda^2 + v^2} \\ -\frac{\lambda}{\lambda^2 + v^2} & -\lambda \end{pmatrix} = 0$$

$$+ x^2 (v^2 + \lambda^2) - \frac{\lambda^2}{\lambda^2 + v^2} = 0$$

$$\Rightarrow \lambda = \pm \frac{\lambda}{\lambda^2 + v^2} \quad \text{are principle curvatures}$$

5 marks

$$\Rightarrow K_G = \left(\frac{\lambda}{\lambda^2 + v^2} \right) \left(-\frac{\lambda}{\lambda^2 + v^2} \right)$$

$$= -\frac{\lambda^2}{(\lambda^2 + v^2)^2}$$

5 marks

$$6. \quad N_\sigma = \frac{\sigma_u \times \sigma_v}{\sqrt{EG - F^2}}$$

3 marks

①

$$\begin{aligned} N_\sigma \times \sigma_u &= \frac{(\sigma_u \times \sigma_v) \times \sigma_u}{\sqrt{EG - F^2}} \\ &= \frac{(\sigma_u \cdot \sigma_u) \sigma_v - (\sigma_u \cdot \sigma_v) \sigma_u}{\sqrt{EG - F^2}} \end{aligned}$$

unseen,
but uses
ideas familiar
↓ them

$$= \frac{E \sigma_v - F \sigma_u}{\sqrt{EG - F^2}}$$

3 marks

$$\begin{aligned} N_\sigma \times \sigma_v &= \frac{(\sigma_u \times \sigma_v) \times \sigma_v}{\sqrt{EG - F^2}} \\ &= \frac{(\sigma_u \cdot \sigma_v) \sigma_v - (\sigma_u \cdot \sigma_v) \sigma_u}{\sqrt{EG - F^2}} \\ &= \frac{F \sigma_v - G \sigma_u}{\sqrt{EG - F^2}} \end{aligned}$$

3 marks

$$b) \quad N \times \dot{\gamma} = N \times (\dot{u} \sigma_u + \dot{v} \sigma_v)$$

$$= \dot{u} (N \times \sigma_u) + \dot{v} (N \times \sigma_v)$$

$$= \frac{\dot{u} (E \sigma_v - F \sigma_u) + \dot{v} (F \sigma_v - G \sigma_u)}{\sqrt{EG - F^2}} \quad \text{by a)}$$

3 marks

c) $\ddot{\gamma} = \frac{d}{dt} (\dot{u} \sigma_u + \dot{v} \sigma_v)$

$$= \ddot{u} \sigma_u + \dot{u} (\sigma_{uu} \dot{u} + \sigma_{uv} \dot{v}) + \ddot{v} \sigma_v + \dot{v} (\sigma_{vu} \dot{u} + \sigma_{vv} \dot{v})$$

$$= \ddot{u} \sigma_u + \ddot{v} \sigma_v + \dot{u}^2 \sigma_{uu} + 2 \sigma_{uv} \dot{u} \dot{v} + \dot{v}^2 \sigma_{vv}$$

(using $\sigma_{uv} = \sigma_{vu}$) 3 marks

d) $\ddot{\gamma} \cdot (N_G \wedge \dot{\gamma})$

$$= (\ddot{u} \sigma_u + \ddot{v} \sigma_v) \cdot \left(\frac{\dot{u} (E \sigma_v - F \sigma_u) + \dot{v} (F \sigma_v - G \sigma_u)}{\sqrt{EG - F^2}} \right)$$
3 marks

$$= \frac{1}{\sqrt{EG - F^2}} \left\{ \begin{aligned} &\ddot{u} \dot{u} \underbrace{(E \sigma_u \cdot \sigma_v - F \sigma_u \cdot \sigma_u)}_{= EF - FE = 0} \\ &+ \ddot{u} \dot{v} \underbrace{(F \sigma_u \cdot \sigma_v - G \sigma_u \cdot \sigma_u)}_{= F^2 - GE} \\ &+ \ddot{v} \dot{u} \underbrace{(E \sigma_v \cdot \sigma_v - F \sigma_v \cdot \sigma_u)}_{= EG - F^2} \\ &+ \ddot{v} \dot{v} \underbrace{(F \sigma_v \cdot \sigma_v - G \sigma_v \cdot \sigma_u)}_{= FG - GF = 0} \end{aligned} \right.$$

5 marks

$$= \frac{1}{\sqrt{EG - F^2}} (EG - F^2) (\ddot{v} \dot{u} - \ddot{u} \dot{v})$$

$$= \sqrt{EG - F^2} (\ddot{v} \dot{u} - \ddot{u} \dot{v}),$$
2 marks